

$\mathbb{Z}_3 \times \mathbb{Z}_3$  CROSSED PRODUCTS

ELIYAHU MATZRI

ABSTRACT. Let  $A$  be the generic abelian crossed product with respect to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ , in this note we show that  $A$  is similar to the tensor product of 4 symbol algebras (3 of degree 9 and one of degree 3) and if  $A$  is of exponent 3 it is similar to the product of 31 symbol algebras of degree 3. We then use [9] to prove that if  $A$  is any algebra of degree 9 then  $A$  is similar to the product of 35840 symbol algebras (8960 of degree 3 and 26880 of degree 9) and if  $A$  is of exponent 3 it is similar to the product of 277760 symbol algebras of degree 3. We then show that the essential 3-dimension of the class of  $A$  is at most 6.

## 1. INTRODUCTION

Throughout this note we let  $F$  be a field containing all necessary roots of unity, denoted  $\rho_n$ . The well known Merkurjev-Suslin theorem says that: assuming  $F$  contains a primitive  $n$ -th root of 1, there is an isomorphism  $\psi : K_2(F)/nK_2(F) \longrightarrow \text{Br}(F)_n$  sending the symbol  $\{a, b\}$  to the symbol algebra  $(a, b)_{n, F}$ . In particular the  $n$ -th torsion part of the Brauer group is generated by symbol algebras of degree  $n$ . This means every  $A \in \text{Br}(F)_n$  is similar (denoted  $\sim$ ) to the tensor product of symbol algebras of degree  $n$ . However, their proof is not constructive. It thus raises the following questions. Let  $A$  be an algebra of degree  $n$  and exponent  $m$ . Can one explicitly write  $A$  as the tensor product of degree  $m$  symbol algebras? Also, what is the smallest number of factors needed to express  $A$  as the tensor product of degree  $m$  symbol algebras? This number is sometimes called the Merkurjev-Suslin number. These questions turn out to be quite hard in general and not much is known. Here is a short summary of some known results.

- (1) Every degree 2 algebra is isomorphic to a quaternion algebra.

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- (2) Every degree 3 algebra is cyclic thus if  $\rho_3 \in F$  it is isomorphic to a symbol algebra (Wedderburn [13]).
- (3) Every degree 4 algebra of exponent 2 is isomorphic to a product of two quaternion algebras (Albert [1]).
- (4) Every degree  $p^n$  symbol algebra of exponent  $p^m$  is similar to the product of  $p^{n-m}$  symbol algebras of degree  $p^m$  (Tignol [11]).
- (5) Every degree 8 algebra of exponent 2 is similar to the product of four quaternion algebras (Tignol [12]).
- (6) Every abelian crossed product with respect to  $\mathbb{Z}_n \times \mathbb{Z}_2$  is similar to the product of a symbol algebra of degree  $2n$  and a quaternion algebra, in particular, due to Albert [1], every degree 4 algebra is similar to the product of a degree 4 symbol algebra and a quaternion algebra (Lorenz, Rowen, Reichstein, Saltman [5]).
- (7) Every abelian crossed product with respect to  $(\mathbb{Z}_2)^4$  of exponent 2 is similar to the product of 18 quaternion algebras (Sivatski [10]).
- (8) Every  $p$ -algebra of degree  $p^n$  and exponent  $p^m$  is similar to the product of  $p^n - 1$  cyclic algebras of degree  $p^m$  (Florence [3]).

In this paper we prove theorems 3.3 and 4.1 stating:

*Let  $A$  be an abelian crossed product with respect to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ . Then*

- (1)  *$A$  is similar to the product of 4 symbol algebras (3 of degree 9 and one of degree 3).*
- (2) *If  $A$  is of exponent 3 then  $A$  is similar to the product of 31 symbol algebras of degree 3.*

We then use [9] to deduce the general case of an algebra of degree 9 to get theorem 5.1 stating:

*Let  $A$  be an  $F$ -central simple algebra of degree 9. Then*

- (1)  *$A$  is similar to the product of 35840 symbol algebras, (8960 of degree 3 and 26880 of degree 9).*
- (2) *If  $A$  is of exponent 3 then  $A$  is similar to the product of 277760 symbol algebras of degree 3.*

## 2. $\mathbb{Z}_p \times \mathbb{Z}_p$ ABELIAN CROSSED PRODUCTS

Let  $A$  be the generic abelian crossed product with respect to  $G = \mathbb{Z}_p \times \mathbb{Z}_p$  over  $F$ , where  $p$  is an odd prime. In the notation of [2] this means:  $A = (E, G, b_1, b_2, u) = E[z_1, z_2 | z_i e z_i^{-1} = \sigma_i(e); z_1^p = b_1; z_2^p = b_2; z_2 z_1 = u z_1 z_2; b_i \in E_i = E^{<\sigma_i>}; u \in E^\times \text{ s.t. } N_{E/F}(u) = 1]$  where  $\text{Gal}(E/F) = \langle \sigma_1, \sigma_2 \rangle \cong G$ .

Let  $A$  be as above. Write  $E = E_1 E_2$  where  $E_1 = F[t_1 | t_1^p = f_1 \in F^\times]$  and  $E_2 = F[t_2 | t_2^p = f_2 \in F^\times]$  thus we have  $z_i t_i z_i^{-1} = \sigma_i(t_i) = t_i$  and  $z_1 t_2 = \rho_p t_2 z_1; z_2 t_1 = \rho_p t_1 z_2$ . Since  $b_i \in E_i$  we can write  $b_1 = c_0 + c_1 t_1 + \dots + c_{p-1} t_1^{p-1}; b_2 = a_0 + a_1 t_2 + \dots + a_{p-1} t_2^{p-1}$  where  $a_i, c_i \in F^\times$ .

**Proposition 2.1.** *Define  $v = e_1 z_1 + e_2 z_2$  for  $e_i \in E$ . If  $v \neq 0$ , then  $[F[v^p] : F] = p$ .*

*Proof.* First we compute  $vt_1 t_2 = (e_1 z_1 + e_2 z_2) t_1 t_2 = e_1 z_1 t_1 t_2 + e_2 z_2 t_1 t_2 = \rho_p t_1 t_2 e_1 z_1 + \rho_p t_1 t_2 e_2 z_2 = \rho_p t_1 t_2 (e_1 z_1 + e_2 z_2) = \rho_p t_1 t_2 v$ . Thus  $v^p$  commutes with  $t_1 t_2$  where  $v$  does not, implying  $[F[v] : F[v^p]] = p$ . By the definition of  $v$  we have  $v \notin F$ . Thus  $\deg(A) = p^2$  imply  $[F[v] : F] \in \{p, p^2\}$ . If  $[F[v] : F] = p$  we get that  $A$  contains the sub-algebra generated by  $t_1 t_2, v$  which is a degree  $p$  symbol over  $F$  and by the double centralizer this will imply that  $A$  is decomposable which is not true in the generic case. Thus  $[F[v] : F] = p^2$  implying  $[F[v^p] : F] = p$  and we are done.  $\square$

The first step we take is to find a  $v$  satisfying  $\text{Tr}(v^p) = 0$ . In order to achieve that we will tensor  $A$  with an  $F$ -symbol of degree  $p$ .

Define  $B = (E_1, \sigma_2, \frac{-c_0}{a_0}) \sim (E, G, 1, \frac{-c_0}{a_0}, 1)$ . Now by [2]  $A \otimes B$  is similar to  $C = (E, G, b_1, \frac{-c_0}{a_0} b_2, u)$ . Abusing notation we write  $z_1, z_2$  for the new ones in  $C$ .

**Proposition 2.2.** *Defining  $v = z_1 + z_2$  in  $C$  we have  $\text{Tr}(v^p) = 0$ .*

*Proof.* First notice that  $C = \sum_{i,j=0}^{p-1} E z_1^i z_2^j$ . Thus  $C_0 = \{d \in C | \text{Tr}(d) = 0\} = E_0 + \sum_{i,j=0; (i,j) \neq (0,0)}^{p-1} E z_1^i z_2^j$  where  $E_0 = \sum_{i,j=0; (i,j) \neq (0,0)}^{p-1} F t_1^i t_2^j$  is the set of trace zero elements of  $E$ . Now computing we see  $v^p = z_1^p + e_{p-1,1} z_1^{p-1} z_2 + \dots + e_{1,p-1} z_1 z_2^{p-1} + z_2^p = b_1 + e_{p-1,1} z_1^{p-1} z_2 + \dots + e_{1,p-1} z_1 z_2^{p-1} + b_2$  where  $e_{i,j} \in E$ . Define  $r = v^p - (b_1 + b_2)$ . Clearly  $\text{Tr}(r) = 0$ , since the powers of  $z_1, z_2$  in all monomial appearing in  $r$  are less than  $p$  and at least one is greater than zero. Thus,  $v^p = b_1 + b_2 + r = c_0 + c_1 t_1 + \dots + c_{p-1} t_1^{p-1} + (-c_0 + \frac{-c_0 a_1}{a_0} t_2 + \dots + \frac{-c_0 a_{p-1}}{a_0} t_2^{p-1}) + r \in C_0$ , and we are done.  $\square$

**Proposition 2.3.**  $K \doteq F[t_1 t_2, v^p]$  is a maximal subfield of  $C$ .

*Proof.* First, notice  $C$  is a division algebra of degree  $p^2$ . To see this assume it is not, then it is similar to a degree  $p$  algebra,  $D$ . Thus  $A \otimes B$  is similar to  $D$ , which implies  $A$  is isomorphic to  $D \otimes B^{op}$ . But then  $A$  has exponent  $p$  which is false. In the proof of 2.1 we saw that  $[v^p, t_1 t_2] = 0$  so we are left with showing  $[K : F] = p^2$ . Assuming  $[K : F] = p$ , we have  $v^p \in F[t_1 t_2]$ . Let  $\sigma$  be a generator of  $\text{Gal}(F[t_1 t_2]/F) = \langle \sigma \rangle$ . Clearly  $z_i x = \sigma(x) z_i$  for  $i = 1, 2$  and  $x \in F[t_1 t_2]$ , hence  $vx = \sigma(x)v$ , that is  $\sigma(x) = vxv^{-1}$ . In particular,  $\sigma(v^p) = vv^p v^{-1} = v^p$ , implying  $v^p \in F$ . But then  $C$  contains the subalgebra  $F[t_1 t_2, v]$  which is an F-csa of degree  $p$ , thus by the double centralizer  $C$  would decompose into two degree  $p$  algebras. This will imply that  $A$  has exponent  $p$ , which is false.  $\square$

The next step is to make  $K$  Galois. Let  $T$  be the Galois closure of  $F[v^p]$ . Its Galois group is a subgroup of  $S_p$  so has a cyclic  $p$ -Sylow subgroup, define  $L$  to be the fixed subfield. Clearly  $F[v^p] \otimes L$  is Galois, with group  $\mathbb{Z}_p$ . Thus in  $C_L$  we have  $K_L$  as a maximal Galois subfield with group  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Now writing  $C_L$  as an abelian crossed product we have  $C_L = (K, G, b_1, b_2, u)$  where this time we have  $\text{Tr}(b_2) = 0$ . Thus we can write  $K_L = L[t_1 t_2, t_3 | (t_1 t_2)^p = f_1 f_2; t_3^p = l \in L]$ ,  $b_1 \in L[t_1 t_2]$  and  $b_2 = l_1 t_3 + \dots + l_{p-1} t_3^{p-1}$ .

Now we change things even more. Define  $D = (f_1 f_2, (-\frac{f_1 f_2}{l_1})^p l^{-1})_{p^2, L} = (K_L, G, t_1 t_2, -\frac{f_1 f_2}{l_1} (t_3)^{-1}, \rho_{p^2})$  and again by [2] we have  $R \doteq C_L \otimes D = (K_L, G, t_1 t_2 b_1, -f_1 f_2 - \frac{f_1 f_2 l_2}{l_1} t_3 - \dots - \frac{f_1 f_2 l_{p-2}}{l_1} t_3^{p-2}, \rho_{p^2} u)$ .

3. GENERIC  $\mathbb{Z}_3 \times \mathbb{Z}_3$  ABELIAN CROSSED PRODUCTS

From now we specialize to  $p = 3$ .

**Proposition 3.1.**  *$R$  from the end of the previous section is a symbol algebra of degree 9.*

*Proof.* This proof is just as in [5]. Since we assume  $\rho_9 \in F$  it is enough to find a 9-central element. Notice that in  $R$  we have  $z_2 t_1 t_2 = \rho_3 t_1 t_2 z_2$ ;  $z_2^3 = -f_1 f_2 - \frac{f_1 f_2 l_2}{l_1} t_3$  and  $(t_1 t_2)^3 = f_1 f_2$ . Thus defining  $x = t_1 t_2 + z_2$  we get  $x^3 = (t_1 t_2 + z_2)^3 = (t_1 t_2)^3 + z_2^3 = -\frac{f_1 f_2 l_2}{l_1} t_3$  implying  $x^9 = -(\frac{f_1 f_2 l_2}{l_1})^3 l \in L$ . Thus  $R = (l_3, -(\frac{f_1 f_2 l_2}{l_1})^3 l)_{9,L}$  for some  $l_3 \in L$  and we are done.  $\square$

All of the above gives the following theorem:

**Theorem 3.2.** *Let  $A$  be a generic abelian crossed product with respect to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ . Then after a quadratic extension  $L/F$  we have  $A_L$  is similar to  $R \otimes D^{-1} \otimes B^{-1}$  where  $R, D, B$  are symbols as above.*

In order to go down to  $F$  we take corestriction. Using Rosset-Tate and the projection formula, ([4] 7.4.11 and 7.2.7), we get:

**Theorem 3.3.** *Let  $A$  be a generic abelian crossed product with respect to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ . Then  $A = \sum_{i=1}^4 C_i$  where  $C_1, C_2, C_3$  are symbols of degree 9 and  $C_4$  is a symbol of degree 3.*

*Proof.* One gets  $C_1, C_2$  from the corestriction of  $R$  using R.T.  $C_3$  from the corestriction of  $D$  using the projection formula and  $C_4$  comes from  $B$ .  $\square$

## 4. THE EXPONENT 3 CASE

In this section we will consider the case where  $\exp(A) = 3$ . Notice that from 3.2  $A_L \sim R \otimes D^{-1} \otimes B^{-1} = (a, b)_{9,L} \otimes (\gamma, c)_{9,L} \otimes (\alpha, \beta)_{3,L}$  where  $\alpha, \beta, \gamma \in F^\times$  and  $a, b, c \in L^\times$ .

**Theorem 4.1.** *Assume  $A$  has exponent 3, then  $A$  is similar to the sum of 16 degree 3 symbols over a quadratic extension and 31 degree 3 symbols over  $F$ .*

*Proof.* The idea for this proof is credited to L.H. Rowen, U. Vishne and E. Matzri. Since  $\exp(A) = 3$  we have  $F \sim A^3 \sim R^3 \otimes D^{-3} \otimes B^{-3} \sim R^3 \otimes D^{-3} \sim (a, b)_{3,L} \otimes (\gamma, c)_{3,L}$ . Thus we get  $(a, b)_{3,L} = (\gamma, c^{-1})_{3,L}$ . Now by the chain lemma for degree 3 symbols in [8] or [6] we have  $x_{1,2,3} \in L^\times$  such that:

$$(a, b)_{3,L} = (a, x_1)_{3,L} = (x_2, x_1)_{3,L} = (x_2, x_3)_{3,L} = (\gamma, x_3)_{3,L} = (\gamma, c^{-1})_{3,L}$$

Now we write

$$(a, \frac{b}{x_1})_{9,L} \otimes (\frac{a}{x_2}, x_1)_{9,L} \otimes (x_2, \frac{x_1}{x_3})_{9,L} \otimes (\frac{x_2}{\gamma}, x_3)_{9,L} \otimes (\gamma, x_3 c)_{9,L} \sim (a, b)_{9,L} \otimes (\gamma, c)_{9,L}$$

Thus  $A \sim (a, b)_{9,L} \otimes (\gamma, c)_{9,L} \otimes (\alpha, \beta)_{3,L} \sim (a, \frac{b}{x_1})_{9,L} \otimes (\frac{a}{x_2}, x_1)_{9,L} \otimes (x_2, \frac{x_1}{x_3})_{9,L} \otimes (\frac{x_2}{\gamma}, x_3)_{9,L} \otimes (\gamma, x_3 c)_{9,L} \sim (a, b)_{9,L} \otimes (\gamma, c)_{9,L} \otimes (\alpha, \beta)_{3,L}$  where now all the degree 9 symbols are of exponent 3. But by a theorem of Tignol, [11], each of these symbols is similar to the product of three degree 3 symbols. Thus we have that  $A_L$  is similar to the product of 16 degree 3 symbols and over  $F$  to the product of 31 symbols of degree 3 and we are done.

□

## 5. THE GENERAL CASE OF A DEGREE 9 ALGEBRA

In this section we combine the results of sections 2 and 3 with [9] to handle the general case of a degree 9 algebra of exponent 9 and 3. Let  $A$  be a  $F$ -central simple algebra of degree 9.

The first step would be to follow [9] to find a field extension  $P/F$  such that  $A_P$  is an abelian crossed product with respect to  $\mathbb{Z}_3 \times \mathbb{Z}_3$  and  $[P : F]$  is prime to 3. The argument in [9] basically goes as follows: Let  $K \subset A$  be a maximal subfield, i.e.  $[K : F] = 9$ . Now let  $F \subset K \subset E$  be the normal closure of  $K$  over  $F$ . Since we know nothing about  $K$  we have to assume  $G = \text{Gal}(E/F) = S_9$ . Let  $H < G$  be a 3-sylow subgroup and  $L = E^H$ , then  $[L : H] = 4480$ . Now extend scalars to  $L$ , then  $KL \subset A_L$  as a maximal subfield. By Galois correspondence  $KL = E^{H_1}$  for some subgroup  $H_1 < H$  and  $[H : H_1] = [KL : L] = 9$ . Since  $H$  is a 3-group we can find  $H_1 \triangleleft H_2 \triangleleft H$  such that  $[H : H_2] = 3$  thus we have  $L = E^H \subset E^{H_2} \subset KL = E^{H_1} \subset E$  and since  $H_2 \triangleleft H$  we know the extension  $E^{H_2}/L$  is Galois with group  $\text{Gal}(E^{H_2}/L) = \langle \sigma \rangle \cong H/H_2 \cong C_3$ . Thus in  $A_L$  we have the subfield  $E^{H_2}$  which has a non trivial  $L$ - automorphism  $\sigma$ . Now let  $z \in A$  be an element inducing  $\sigma$  (such  $z$  exists by Skolem-Noether). Consider the subfield  $L[z]/L$ , since  $z^3$  commutes with  $E^{H_2}$  and  $z$  does not  $[L[z] : L[z^3]] = 3$ . In the best case scenario we have  $L[z^3] = L$  which will imply  $A_L$  decomposes into the tensor product of two symbols of degree 3 and we are done. In the general case we will have  $[L[z^3] : L] = 3$ . If  $L[z^3]/L$  is Galois we are done since  $E^{H_2}[z^3]$  will be a maximal subfield Galois over  $L$  with group isomorphic to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ , but again in general this should not be the case. However we can extend scalars to make  $L[z^3]/L$  Galois, in particular consider  $P = L[\text{disc}(L[z^3])]$  then,  $[P : L] = 2$  and  $P[z^3]/P$  is Galois and we are done. To summarize we have found an extension  $P = L[\text{disc}(L[z^3])]$  with  $[P : F] = 4480 \cdot 2 = 8960$  such that  $A_P$  contains a maximal subfield  $PE^{H_2}[z^3]/P$  Galois over  $P$  with group isomorphic to  $\mathbb{Z}_3 \times \mathbb{Z}_3$ .

Combining the above with the results of sections 2,3 and using Rosset-Tate we get the following theorem.

**Theorem 5.1.** *Let  $A$  be an  $F$ -central simple algebra of degree 9. Then*

- (1)  *$A$  is similar to the product of 35840 symbol algebras, (8960 of degree 3 and 26880 of degree 9).*
- (2) *If  $A$  is of exponent 3 then  $A$  is similar to the product of 277760 symbol algebras of degree 3.*

## 6. APPLICATION TO ESSENTIAL DIMENSION

In [7] Merkurjev computes the essential  $p$ -dimension of  $PGL_{p^2}$  relative to a fixed field  $k$  to be  $p^2 + 1$ . One can interpret this result as follows: Let  $F$  be a field of definition (relative to a base field  $k$ ) for the generic division algebra of degree  $p^2$ . Let  $E/F$  be the prime to  $p$  closure of  $F$ . Let  $l, l \subset k \subset E$ , be a subfield of  $E$  over which  $A$  is defined. Then  $l/k$  has transcendence degree at least  $p^2 + 1$  (and such  $l$  exists with transcendence degree exactly  $p^2 + 1$ ). It makes sense to define the essential dimension and the essential  $p$ -dimension of the class of an algebra  $A$  (with respect to a fixed base field  $k$ ).

**Definition 6.1.** *Let  $A \in \text{Br}(F)$ . Define the essential dimension and the essential  $p$ -dimension of the class of  $A$  (with respect to a fixed base field  $k$ ) as:*

$$\begin{aligned} \text{edc}(A) &= \min\{\text{ed}(B) | B \sim A\} \\ \text{edc}_p(A) &= \min\{\text{ed}_p(B) | B \sim A\} \end{aligned}$$

Notice that [7] for  $p=2$  gives  $\text{ed}_2(PGL_{2^2}) = 5$  and for  $p = 3$  it gives  $\text{ed}_3(PGL_{3^2}) = 10$ . Now assume  $F$  is prime to  $p$  closed. Then as proved in [9] every  $F$ -csa of degree  $p^2$  is actually an abelian crossed product with respect to  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Thus, in this language, in [5] they prove:

**Theorem 6.2.** *Let  $A$  be a generic division algebra of degree 4, then  $\text{edc}(A) = \text{edc}_2(A) = 4$*

For  $p = 3$  Theorem 3.3 says:

**Theorem 6.3.** *Let  $A$  be a generic division algebra of degree 9, then  $\text{edc}_3(A) \leq 6$*



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